

Comment on “Higher order Painlevé equations and their symmetries via reductions of a class of integrable models”

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This comment is written to compare the results of our two papers [8] and [9] with the results of [1]. In [8] we have been proposed a modified version of Krichever-Dickey rational reductions of the KP hierarchy [3], [5] with corresponding lattice representation. To make our comment more clear, let us first give a short description of this approach.

Given any positive integers $n \geq 1$ and $p > n$, corresponding integrable hierarchy is defined by two evolution equations

$$\partial_s G_{(k)} = (\mathcal{L}_{(k+1)}^s)_+ G_{(k)} - G_{(k)} (\mathcal{L}_{(k)}^s)_+, \quad (0.1)$$

$$\partial_s H_{(k)} = (\mathcal{M}_{(k+1)}^s)_+ H_{(k)} - H_{(k)} (\mathcal{M}_{(k)}^s)_+ \quad (0.2)$$

with

$$G_{(k)} \equiv \partial + v_k, \quad H_{(k)} \equiv \partial + u_k$$

and finite collection of pseudo-differential operators $\{\mathcal{L}_{(j)}, \mathcal{M}_{(j)}\}$ defined through

$$\mathcal{L}_{(j)}^{p-n} = G_{(j-1)} \cdots G_{(1)} H_{(1)}^{-1} \cdots H_{(n)}^{-1} G_{(p)} \cdots G_{(j)}, \quad j = 1, \dots, p+1,$$

$$\mathcal{M}_{(j)}^{p-n} = H_{(j)}^{-1} \cdots H_{(n)}^{-1} G_{(p)} \cdots G_{(1)} H_{(1)}^{-1} \cdots H_{(j-1)}^{-1}, \quad j = 1, \dots, n+1.$$

As a consequence of (0.1) and (0.2), pseudo-differential operators $L = P_n^{-1} Q_p$ and $M = Q_p P_n^{-1}$ with factorized operators

$$Q_p = G_{(p)} \cdots G_{(1)}, \quad P_n = H_{(n)} \cdots H_{(1)}$$

satisfy KP Lax equations. The fields v_j and u_j , by virtue of their definition, must satisfy the condition

$$\sum_{j=1}^p v_j = \sum_{j=1}^n u_j. \quad (0.3)$$

This means that in fact one has $n+p-1$ independent fields. Let us also write down evolution equations governing the second flow in the hierarchy in explicit form. We get from (0.1) and (0.2) that

$$\partial_2 v_k = (v'_k - v_k^2)' + \frac{2}{p-n} \left(\sum_{j=1}^{n-1} (n-j) u'_j - \sum_{j=1}^{k-1} (n-j) v'_j - \sum_{j=k}^{p-1} (p-j) v'_j \right)' - R', \quad (0.4)$$

$$\partial_2 u_k = (u'_k - u_k^2)' + \frac{2}{p-n} \left(-\sum_{j=1}^{p-1} (p-j)v'_j + \sum_{j=1}^{k-1} (p-j)u'_j + \sum_{j=k}^{n-1} (n-j)u'_j \right)' - R' \quad (0.5)$$

with

$$R \equiv \frac{2}{p-n} \left(\sum_{j < k} v_j v_k - \sum_{j < k} u_j u_k \right).$$

In the paper [1], the authors consider the particular case $p = n + 1$ with $n = M$ of modified rational reductions of KP hierarchy given by (0.1) and (0.2). Remark that the second flow in this case is yielding by the system of evolution equations

$$\partial_2 v_k = (v'_k - v_k^2)' + 2 \left(\sum_{j=1}^{n-1} (n-j)u'_j - \sum_{j=1}^{k-1} (n-j)v'_j - \sum_{j=k}^n (n-j+1)v'_j \right)' - R',$$

$$\partial_2 u_k = (u'_k - u_k^2)' + 2 \left(-\sum_{j=1}^n (n-j+1)v'_j + \sum_{j=1}^{k-1} (n-j+1)u'_j + \sum_{j=k}^{n-1} (n-j)u'_j \right)' - R'$$

which is a specification of (0.4) and (0.5). Clearly, in this case one has $2M$ independent fields. Lax operator (2.10) denoted by L_M in [1] is of the form $Q_{M+1}P_M^{-1}$ with factorized operators P_M and Q_{M+1} and it involves a finite number of the fields $\{c_j, e_j : k = j, \dots, M\}$ in such a way that our fields v_k and u_k turn out linearly depend on $\{c_j, e_j\}$ and moreover the condition (0.3) becomes an identity and can be discarded. For example, in the case $M = 2$, it can be computed to obtain

$$v_1 = -c_1 - c_2, \quad v_1 = -e_1 - c_2, \quad v_1 = -e_2,$$

$$u_1 = -e_1 - c_1 - c_2, \quad v_1 = -e_2 - c_2.$$

The authors of [1] write down the second flow in explicit form (3.1) making use the corresponding Hamiltonian coming from Lax operator. One has only to note that their equations differ from our equations in sign of right-hand sides. As for discrete symmetry, transformation g in [1] is in fact discrete symmetry transformation s_1 in [8] which in general case is generated by differential-difference equation

$$\sum_{j=1}^{p-n} a'_{i+j-1} = \sum_{j=1}^{p-n} a_{i+j-1} \left(\sum_{j=1}^n a_{i+j-n-1} - \sum_{j=1}^n a_{i+j+p-n-1} \right) \quad (0.6)$$

and responsible for the shift $i \rightarrow i + n$. In the particular case $p = n + 1$, in an obvious way, equation (0.6) becomes well-known Itoh-Narita-Bogoyavlenskii (INB) lattice [4], [6], [2]

$$a'_i = a_i \left(\sum_{j=1}^n a_{i-j} - \sum_{j=1}^n a_{i+j} \right) \quad (0.7)$$

which governs discrete symmetries of corresponding integrable hierarchies of evolution differential equations.

In section 6 of [9], we have been shown that self-similarity reduction of INB lattice hierarchy leads to Painlevé equations of type $A_{2n}^{(1)}$. Simplifying the situation, we may consider INB lattice (0.7) supplemented by the condition

$$a_i \left(\sum_{j=-n}^n a_{i+j} - x \right) = \alpha_i, \quad \sum_{j=1}^n \alpha_{i-j} = \sum_{j=1}^n \alpha_{i+j} + 1 \quad (0.8)$$

which follows in fact from self-similarity constraint $(1 + x\partial + 2t_2\partial_2) a_i = 0$. As was shown in [8], that stationary version of (0.8), for all $n \geq 1$, is an integrable discretization of the P_I equation $w'' = 6w^2 + t$. In subsection 4.2 of the work [9], it was shown that INB lattice (0.7) together with compatible constraint (0.8) is equivalent to Painlevé equations of type $A_{2n}^{(1)}$ [7]. It is natural that the authors of [1] using the self-similarity reduction of their evolution equations also come to $A_{2n}^{(1)}$ Painlevé equations.

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